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Square lattice Ising model $\tilde{\chi}^{(5)}$ ODE in exact arithmetic

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Abstract
We obtain in exact arithmetic the order 24 linear differential operator L_{24} and the right-hand side $E^{(5)}$ of the inhomogeneous equation $L_{24}(\Phi^{(5)}) = E^{(5)}$, where $\Phi^{(5)} = \tilde{\chi}^{(5)} - \tilde{\chi}^{(3)}/2 + \tilde{\chi}^{(1)}/120$ is a linear combination of n-particle contributions to the susceptibility of the square lattice Ising model. In Bostan *et al* (2009 *J. Phys. A: Math. Theor.* **42** 275209), the operator L_{24} (modulo a prime) was shown to factorize into $L_{12}^{(\mathrm{left})} \cdot L_{12}^{(\mathrm{right})}$; here we prove that no further factorization of the order 12 operator $L_{12}^{(\mathrm{left})}$ is possible. We use the exact ODE to obtain the behaviour of $\tilde{\chi}^{(5)}$ at the ferromagnetic critical point and to obtain a limited number of analytic continuations of $\tilde{\chi}^{(5)}$ beyond the principal disc defined by its high temperature series. Contrary to a speculation in Boukraa *et al* (2008 *J. Phys. A: Math. Theor.* **41** 455202), we find that $\tilde{\chi}^{(5)}$ is singular at $w = 1/2$ on an infinite number of branches.

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1. Introduction

The story of the zero-field susceptibility χ of the two-dimensional Ising model is a landmark saga of mathematical physics. A recent review of the highlights can be found in [1]. While a closed form expression for the susceptibility still eludes us, we possess an enormous amount of exact or extremely precise numerical information. This largely derives from two complementary approaches. One approach involves studying the series expansion of the susceptibility. Since the work of Orrick *et al* [2] we have had available a polynomial time algorithm, now of complexity $O(N^4)$ for a series of length N terms. From the point of view of an algebraic combinatorialist, this comprises a solution, and many questions about the asymptotics, and about the scaling functions, have been answered by the analysis of the very long series we now have available—currently more than two thousand terms in length.

The difficulty in proceeding further with this approach is that we have no idea what the underlying closed-form solution looks like, except that it is known, or more precisely universally believed, to be non-holonomic. The alternative approach is to express the susceptibility as a form-factor expansion. This approach was initiated more than 30 years ago by Wu *et al* [3]. In this representation, the susceptibility is written as

$$k_B T\,\chi = \frac{1}{s}(1-s^4)^{1/4}\sum_{n\geqslant 0}\tilde{\chi}^{(2n+1)} \tag{1}$$

for $T > T_c$, where $s = \sinh\,(J/k_BT)$. For $T < T_c$ a similar expression with even superscripts prevails. The advantage of the form-factor approach is that each term in the sum *is* holonomic. This means that, with sufficient computational resources, and sufficient ingenuity, each term can be found. To date, the first six terms have been found, in the sense that their defining ODE has been obtained, either totally or *modulo* a prime. We also have precise integral representations for the form-factor terms, and a Landau analysis of the integrands can provide information as to the distribution of singularities in the complex plane. Indeed, it was just such a study by Nickel [4, 5] that gave convincing evidence of a natural boundary in the *total* susceptibility, thus supporting an earlier but weaker argument of Guttmann and Enting [6] that the total susceptibility was non-holonomic.

While an exact solution for the Ising model susceptibility may be impossible and is certainly beyond reach at present, one might hope to obtain a complete picture of the singularities of the susceptibility. Indeed, this more limited goal has been the main motivation of the recent studies of the individual form-factor terms. For this the ODE and Landau analysis approaches are complementary; the Landau analysis provides necessary but not sufficient conditions [7] while the ODE, even if only in modulo a prime representation, can show which Landau singularities are to be excluded. The most detailed study in this regard is that of the five particle contribution $\tilde{\chi}^{(5)}$ to the susceptibility initiated by Boukraa *et al* [8] and followed by Bostan *et al* [9]. The present paper is an attempt to address a number of issues left unresolved in these papers.

A brief summary of the parts of [8, 9] relevant here is as follows. In Boukraa *et al* [8] series in w modulo a prime to 10 000 terms were given and shown to be adequate to find the order 33 Fuchsian differential equation[6], modulo a prime, $L_{33}(\tilde{\chi}^{(5)}) = 0$. Subsequently in Bostan *et al* [9], the complexity of this ODE was shown to be reducible to an inhomogeneous equation

$$L_{24}(\Phi^{(5)}) = E^{(5)}, \tag{2}$$

where $\Phi^{(5)}$ is the linear combination of 5-, 3- and 1-particle contributions:

$$\Phi^{(5)} = \tilde{\chi}^{(5)} - \tilde{\chi}^{(3)}/2 + \tilde{\chi}^{(1)}/120. \tag{3}$$

[6] The notation used here is that in [8], variables w and s are useful for high-temperature expansions with $w = s/2/(1+s^2)$. The high-temperature $\chi^{(2n+1)}$ and $\tilde{\chi}^{(2n+1)}$ are related by $s \cdot \chi^{(2n+1)} = (1-s^4)^{1/4} \cdot \tilde{\chi}^{(2n+1)}$ so that $\tilde{\chi}^{(2n+1)}$ has the 'simpler' divergence $\propto 1/(1-4w) \propto 1/(1-s)^2$ at the ferromagnetic critical point.

The right-hand side of (2) which satisfies $L_5(E^{(5)}) = 0$ is of the form

$$E^{(5)} = w \cdot [(1 - 16w^2)^3 \cdot P_{4,0} \cdot K^4 + (1 - 16w^2)^2 \cdot P_{3,1} \cdot K^3 E + (1 - 16w^2) \cdot P_{2,2} \cdot K^2 \cdot E^2 + P_{1,3} \cdot K \cdot E^3 + P_{0,4} \cdot E^4]/(1 + 4w)^6/(1 - 16w^2)^{\kappa}, \quad (4)$$

where $K = K(4w)$ and $E = E(4w)$ are complete elliptic integrals and the $P_{i,j} = P_{i,j}(w)$ are polynomials. The degree of the polynomials and the denominator power κ in (4) depend on the representation of L_{24} in (2). In the case that L_{24} is minimum order 24, L_{24} is of degree 888, while the $P_{i,j}$ are then of degree (at most) 904 with $\kappa = 8$. In [9] results were only reported for a non-minimum order representation modulo a prime. Furthermore it was shown that L_{24} could be factored into $L_{12}^{(\mathrm{left})} L_{12}^{(\mathrm{right})}$ with $L_{12}^{(\mathrm{right})}$ being reducible into several smaller factors, all but one known in exact arithmetic. The question of whether $L_{12}^{(\mathrm{left})}$ could be factored was left unresolved.

The results reported in [8, 9] are an impressive example of what can be obtained by calculation modulo a prime. However there are limitations. Knowing L_{24} and $E^{(5)}$ modulo a prime enables one to deduce only possible local singularities of $\tilde{\chi}^{(5)}$ and not its global behaviour. For example, one cannot determine the amplitudes of the singularities in $\tilde{\chi}^{(5)}$ at the ferromagnetic point at $s = 1$. The leading and first correction term amplitudes were estimated in [8] but this was based on an analysis of the 2000 term exact integer series that had been obtained from multiple modulo prime series by the Chinese remainder theorem.

In the present paper we determine the minimum order L_{24} and $E^{(5)}$ in exact arithmetic. The results can be found on the website [10]. We have also constructed the minimum order exact L_{29} defined by $L_{29}(\Phi^{(5)}) = 0$ from these results. We have not actually needed this operator but provide it nevertheless for those who might find it of interest and in addition we also report in [10] the exact integer coefficients of $\tilde{\chi}^{(5)}$ to 8000 terms that we generated directly[7] from (2). The global information provided by (2) enables us to prove that $L_{12}^{(\mathrm{left})}$ defined by $L_{24} = L_{12}^{(\mathrm{left})} L_{12}^{(\mathrm{right})}$ cannot be factored. We confirm the 500 digit amplitude of the leading ferromagnetic singularity of $\tilde{\chi}^{(5)}$ reported by Bailey *et al* [11] and also some $|s| = 1$ circle singularity amplitudes derived by Nickel [4, 5].

An important question left unresolved in [8] was whether the singularity of the ODE for $\tilde{\chi}^{(5)}$ at the zero of the head polynomial at $w = 1/2$ was in fact a singularity of $\tilde{\chi}^{(5)}$ on some branch of the function. Appendix D in [8] provided an example for which not all the zeros of the head polynomial of the ODE[8] satisfied by an integral were Landau singularities of the integral. However, the integral for $\tilde{\chi}^{(5)}$ was deemed too complicated in [8] to perform a complete Landau singularity analysis leaving only the conclusion that $w = 1/2$ was very likely not a Landau singularity of $\tilde{\chi}^{(5)}$. Here we perform some analytic continuation of $\tilde{\chi}^{(5)}$ beyond the principal disc $|s| \leqslant 1$ and onto other branches. Our exploration, while not exhaustive, is complete enough to show $\tilde{\chi}^{(5)}$ has the singular behaviour $(1 - 2w)^{7/2}$ with non-vanishing amplitude on an infinite number of branches. We have not made any progress in identifying the Landau integrand singularities that give rise to this behaviour.

In section 2 we describe briefly how we can combine results reported in [9] with multiple modulo prime $\Phi^{(5)}$ series of length at most 4000 terms to obtain the exact L_{24} and $E^{(5)}$ in (2). This allows us to determine the series expansion of $\tilde{\chi}^{(5)}$ at the ferromagnetic point by numerical matching of solutions. The results are reported in appendix B. Section 3 is the outline of the proof that $L_{12}^{(\mathrm{left})}$ defined by $L_{24} = L_{12}^{(\mathrm{left})} \cdot L_{12}^{(\mathrm{right})}$ cannot be factored. Finally in section 4

[7] For the required $\tilde{\chi}^{(3)}$ see appendix B and references therein.

[8] We explicitly exclude zeros associated with apparent singularities.

we describe the analytic continuation of $\Phi^{(5)}$ we have performed to obtain information on the behaviour of $\tilde{\chi}^{(5)}$ at $w = 1/2$.

2. The ODE for $\Phi^{(5)}$ in exact arithmetic

We are looking for the unique minimum order L_{24} and associated $E^{(5)}$ satisfying (2) for a given $\Phi^{(5)}$. This is to be done modulo a prime for enough primes[9] that the exact integer L_{24} and $E^{(5)}$ can be reconstructed by the Chinese remainder theorem. To generate the required $\Phi^{(5)}$ in (3) directly from the defining $\tilde{\chi}^{(n)}$ integrals is impractical as the minimum-order L_{24} is of degree 888 implying $25 \times 889 = 22\,225$ unknown coefficients in L_{24}. Added to this are the five polynomials of degree 904 in $E^{(5)}$, i.e. another $5 \times 905 = 4525$ terms. Finding all coefficients is a straightforward linear algebra problem but requires that we have each $\Phi^{(5)}$ modulo prime series to about 26 800 terms[10].

The operator L defined by $L(\phi) = 0$ for a given ϕ is not unique if one does not require L to be of minimum order. The advantage of seeking a non-minimal order L is that the number of unknown coefficients to be found can drop dramatically. For example, only about 6200 terms are needed to obtain the non-minimal L_{29} of order 51 and degree 118 used in [9]. The analogous effect occurs for the inhomogeneous equation (2) and we find that $\Phi^{(5)}$ series of length only about 5300 terms are needed when L_{24} is chosen of order 42 and degree 103 leading to polynomials of degree 155 in $E^{(5)}$ in (4) with now $\kappa = 26$. Note, however, as observed in [8], that the integer coefficients in non-minimal order operators can be outrageously large and we expect that Chinese remainder reconstruction of the non-minimal order L_{24} would be nearly hopeless. Instead, the utility of a modulo prime non-minimal L_{24} lies in its use as a recursion device to extend directly generated (short) $\Phi^{(5)}$ series to series of sufficient length, i.e. 26 800 terms, so that the minimum-order L_{24} and associated $E^{(5)}$ can be found. Such extension requires completely negligible computer resources.

A useful variant of the above approach is to use the non-minimal L_{24} found as described above to generate a long series for a solution S_{24} satisfying $L_{24}(S_{24}) = 0$. This series need only be about 22 300 terms, long enough to enable reconstruction of the minimum order L_{24}. Finding the coefficients in $E^{(5)}$ from $L_{24}(\Phi^{(5)})$ is then a separate and simpler problem.

A further reduction in the length of the $\Phi^{(5)}$ series to be directly generated can be obtained if one knows a factorization of L_{24} with the right division operator in exact arithmetic. This situation can be realized given the four modulo prime series reported in [9]. We have $L_{24} = L_{12}^{(\text{left})} \cdot L_{12}^{(\text{right})}$. Knowing $L_{12}^{(\text{right})}$ in exact arithmetic allows one to obtain any non-minimal order representation of $L_{12}^{(\text{right})}$ modulo any prime[11] and then $\Psi = L_{12}^{(\text{right})}(\Phi^{(5)})$ modulo a prime from the directly generated $\Phi^{(5)}$. If we choose our representation of the known $L_{12}^{(\text{right})}$ as order 18 and degree 42, and the unknown $L_{12}^{(\text{left})}$ as order 32 and degree 89, then the polynomials in $E^{(5)} = L_{12}^{(\text{left})}(\Psi)$ are of degree 199. This gives $33 \times 90 + 5 \times 200 = 3970$ unknown coefficients in $E^{(5)}$ and $L_{12}^{(\text{left})}$ to be determined implying that 4000 terms of the directly generated $\chi^{(5)}$ series is more than adequate. Since the series generation of $\chi^{(5)}$ described in [8] is an $O(N^4 \cdot \ln(N))$ process, this represents a 3-fold reduction in computer time from that needed using an unfactored L_{24} or a 6-fold reduction relative to the unfactored L_{29} approach.

[9] We have found that about 90 primes $p < 2^{15}$ are sufficient.

[10] An alternative to (2) is $L_{29}(\Phi^{(5)}) = 0$ but this requires $\Phi^{(5)}$ series to $30 \times 1238 = 37\,140$ terms. The change of our ODE problem from homogeneous to inhomogeneous form is a substantial reduction in complexity.

[11] This is most easily done by using the minimum order $L_{12}^{(\text{right})}$ to generate a series solution S_{12} which satisfies $L_{12}^{(\text{right})}(S_{12}) = 0$ but does not satisfy $L_n(S_{12}) = 0$ for any $n < 12$. The non-minimal order $L_{12}^{(\text{right})}$ is then found from the series using, for example, the matrix code described in section 3 of [8].

We have generated the $\tilde{\chi}^{(5)}$ series to a minimum $O(w^{4000})$ for 90 primes $p < 2^{15}$ and from these generated L_{24} modulo a prime as described above. To obtain the exact L_{24} is then a problem of rational reconstruction but it is relatively easy from just a few terms to guess a normalization factor[12] that converts the problem to integer reconstruction by the Chinese remainder theorem. The integer coefficients in L_{24} are observed to typically have very large powers of 2 as factors which one can determine by a process of trial division by 2^k. If k is chosen too small, Chinese remainder reconstruction with a fixed number of primes might fail because the unknown coefficient is too large while if k is chosen too large there is failure because the coefficient is no longer an integer. With 84 primes we find an intermediate k range that yields a consistent integer reconstruction for every coefficient in L_{24}. With 90 primes we have a large number of consistency checks that leave no doubt that our reconstruction is exact. We have also confirmed that the apparent singularity constraint equations (A.8) in [8] are satisfied by our reconstructed L_{24} in all cases, that is, 19 849 satisfied conditions on 22 202 non-vanishing coefficients in L_{24}.

In view of the 'massive' calculations required to find L_{24} in exact arithmetic it is natural to ask for some further mathematical and numerical checks of the correctness of the operators L_{24} and $L_{12}^{(\mathrm{left})}$. First of all we have checked directly that $L_{12}^{(\mathrm{right})}$ does indeed right divide L_{24} in exact arithmetic. Secondly, we have confirmed that the exponents of the operators L_{24} and $L_{12}^{(\mathrm{left})}$ are *rational numbers* and in agreement with our previous massive numerical calculations [8]. Thirdly, the minimal order operator L_{29} corresponds to an integral of an algebraic integrand, and it is therefore, as mathematicians say, a *Period* ('Derived From Geometry' [12]): L_{29} is thus, necessarily, *globally nilpotent*. This is a stronger constraint than being a Fuchsian operator (with integer coefficients) having only rational exponents. Since L_{24} is a factor of L_{29} it must also be globally nilpotent, and likewise the left factor $L_{12}^{(\mathrm{left})}$ must be globally nilpotent. We have verified that L_{24}, $L_{12}^{(\mathrm{left})}$ and $L_{12}^{(\mathrm{right})}$[13] are consistent with globally nilpotent operators. This is a very strong indication that our exact expressions for L_{24} and $L_{12}^{(\mathrm{left})}$ are correct. To check global nilpotence numerically requires one to calculate the p-curvature and check that it is zero for almost all primes. In practice one can obviously only do this for the first few primes. The primes used for L_{12} were those smaller than 30 while primes less than 10 were used for L_{24}.

2.1. Computational details

As shown in [8] the calculation of a series for $\tilde{\chi}^{(5)}$ is a problem with computational complexity $O(N^4 \ln N)$. In [8] we initially calculated $\tilde{\chi}^{(5)}$ to 10 000 terms which required some 17 000 CPU hours on an SGI Altrix cluster with 1.6 GHz Itanium2 processors. From this single series we could already exactly identify a simple right divisor of L_{29}, and using this factor we were able to find a solution modulo a second prime using a series of 'just' 5600 terms. Expanding the series to order 5600 took around 1560 CPU hours on the Altrix cluster. The series modulo these two primes then sufficed to find a larger right divisor of L_{29} in exact arithmetic, and using this factor we found that only 4800 terms would be required to find solutions for any subsequent primes.

Shortly after these developments a new system was installed by the National Computational Infrastructure (NCI) whose National Facility provides the national peak computing facility for Australian researchers. This new system is an SGI XE cluster using

[12] This becomes the value of the head polynomial at $w = 0$ and is $2^{12} \cdot 3^{13} \cdot 5^7 \cdot 7^6 \cdot 11^4 \cdot 23 \cdot 29 \cdot 7225\,564\,279 = 4235\,287\,273\,136\,998\,077\,435\,560\,752\,320\,000\,000$.

[13] However, this operator is automatically globally nilpotent courtesy of its direct sum construction.

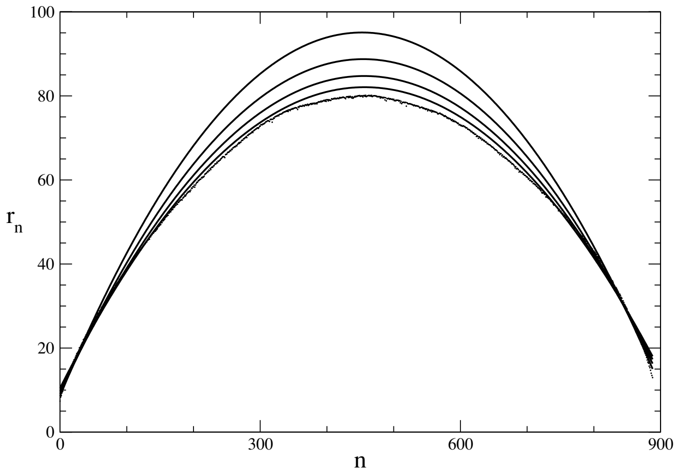

Figure 1. Estimates $r_n = \ln(c_n)/\ln(30\,000)$ for the number of primes required to reconstruct the coefficients c_n.

quad-core 3.0 GHz Intel Harpertown CPUs. Our code runs almost twice as fast on this facility compared to the Altrix cluster. We then used this system to calculate the series for $\tilde{\chi}^{(5)}$ to order 4800 (this took about 450 CPU hours) for a third prime, which again allowed us to find an even larger right divisor of L_{29} in exact arithmetic. This larger operator reduced the required number of terms to 4600 and we then calculated a series for a fourth prime to this order using some 380 CPU hours. These calculations gave us results for four different primes and allowed us to reconstruct the factor $L_{12}^{(\mathrm{right})}$ in exact arithmetic as begun in [9] and completed here in appendix A.

It was only after this that we realized that the inhomogeneous equation (2) could be used, as detailed above, to obtain simultaneous solutions for L_{24} and $E^{(5)}$ using as few as 4000 terms. We then calculated a series for $\tilde{\chi}^{(5)}$ to order 4000 for a further 86 primes with each prime requiring about 215 CPU hours.

The above timings make it clear that the reconstruction of L_{24} and $E^{(5)}$ is a computationally expensive project. The main computational effort is the direct calculation of the series $\tilde{\chi}^{(5)}$ modulo the required number of primes. It is therefore of some practical interest to estimate the number of primes required for the exact reconstruction based only on some partial reconstruction. We focus here on the coefficients of the head-polynomial of L_{24} and denote by c_n the nth coefficient ($n = 0, \ldots, 888$) after stripping it of any factor of 2 as mentioned above. Then $r_n = \ln(c_n)/\ln(30\,000)$ is a rough measure of the number of primes needed to reconstruct c_n. From our previous reconstruction of $L_{12}^{(\mathrm{right})}$ we noticed that the corresponding r_n are given roughly by a quadratic function of n. Thus one can estimate the number of primes needed to reconstruct the full L_{24} from a partial reconstruction since r_n for n near 0 or 888 can be obtained using many fewer primes. In figure 1 the lower 'curve' is the actual data from the coefficients of the head-polynomial of L_{24}, while the upper solid curves are quadratic fits to the data based on the part data set, from top to bottom, $r_n \leqslant 30$, 40, 50 and 60. The pertinent point being that after doing the calculation for some 30 primes we estimated that the reconstruction was likely to succeed with no more than 100 primes and was therefore achievable in practice.

With L_{24} and $E^{(5)}$ known in exact arithmetic, it is easy to use (2) to calculate the exact series for $\Phi^{(5)}$ to high order. Specifically our reconstruction means that we know the coefficients $a_{i,j}$ of the polynomials in the operator L_{24} and the coefficients of the polynomials $P_{i,j}$ of (4) exactly. The latter allows us to easily calculate the coefficients e_n of $E^{(5)}$ by using the (simple) recursive formulae for the elliptic integrals E and K. From (2) we have explicitly, by equating the coefficients of x^n, that (recall that L_{24} is expressed in terms of the differential operator $x\frac{\mathrm{d}}{\mathrm{d}x}$)

$$\sum_{i=0}^{M}\sum_{j=0}^{D} a_{i,j}(n-j)^i c_{n-j} = Ac_n + B = e_n, \tag{5}$$

where A and B are integers (depending on n). The coefficients c_n of $\Phi^{(5)}$ can thus be calculated recursively and from (3) we can calculate the coefficients of $\tilde{\chi}^{(5)}$ (with the coefficients of $\tilde{\chi}^{(3)}$ calculated using the ODE from [13]). We have calculated the coefficients of $\tilde{\chi}^{(5)}$ up to order 8000 and they can be found in [10].

Finally we decided to calculate the minimal order operator L_{29} explicitly; it is given implicitly by (2). This can be done in a variety of ways. The most obvious way is to use (5) to extend the series for $\Phi^{(5)}$ to high enough order ($30 \times 1238 = 37\,140$, since the minimal order L_{29} has degree 1237) and then use the matrix code of [8] to calculate the ODE corresponding to L_{29} modulo a sufficient number of primes to reconstruct L_{29}. However, computationally it is easier to first calculate modulo a prime the minimal operator L_5 annihilating $E^{(5)}$ and then form the product $L_5 \cdot L_{24}$ modulo a prime. The minimal order operator for L_5 has degree 4489 so $6 \times 4490 = 26\,490$ terms of $E^{(5)}$ are required. Obviously, since the minimal order L_{24} has degree 888, the product $L_5 \cdot L_{24}$ has degree 5377 meaning that there is a common factor of degree 4140, which we must discard in order to calculate the degree 1237 polynomials of L_{29}. The minimal order L_5 was calculated (for each prime) using the matrix code of [8]. The product $L_5 \cdot L_{24}$ was then calculated modulo a prime using Maple and the common factor can then be divided out modulo a prime. The bottle neck in this calculation is the use of the matrix code of [8] which has computational complexity $O(N^3)$. It was for this reason that we chose the 'indirect' route of going through L_5 to get L_{29}.

3. Proof that $L_{12}^{(\text{left})}$ does not factorize

The factorization of L_{29} defined by $L_{29}(\Phi^{(5)}) = 0$ described in [9] relied mostly on the testing of series $S(w)$ generated by $L_{29}(S) = 0$ modulo a prime around $w = 0$. If a particular series is annihilated by an L_n for order $n < 29$ then L_n right divides L_{29}. Depending on the singularity exponents, a series solution[14] $S(w) = w^q \cdot (1 + \alpha_1 w + \alpha_2 w^2 + \cdots)$ with fixed q might be uniquely determined by $L_{29}(S) = 0$ or might contain one or more arbitrary rational coefficients α_i. In the latter case, the series may be a generator for a right division operator only for a particular choice of constants α_i and the problem then is how these particular values might be found. If there is only one arbitrary coefficient α in the series $S(w)$, an exhaustive search is possible in a modulo prime, p, calculation because one need then only test p separate series with α an integer satisfying $0 \leqslant \alpha < p$. If there is more than one arbitrary α_i in the series $S(w)$, such brute force 'guessing' is no longer practical due to computational time constraints. It is this that prevented the authors of [9] from deciding whether $L_{12}^{(\text{left})}$ is factorizable. Let us also note that we were not able to perform a straight formal calculation factorization of $L_{12}^{(\text{left})}$

[14] If the solution contains powers of $\ln(w)$, the $S(w)$ here is to be interpreted as the coefficient of the highest power of $\ln(w)$.

using its expression in exact arithmetic and our attempted calculations failed on a computer with 48 Gb of memory[15].

As pointed out in [9] there is nothing special about the singular point $w = 0$. And indeed, a series solution about a singular point $w_s \neq 0$ might also lead to a right division operator. For example, the unique singular series

$$x^{-7/4} \cdot (1 + 387x/80 + 72\,103x^2/23\,040 + 2054\,561x^3/1597\,440 + O(x^4)) \qquad (6)$$

with $x = w - 1/4$ is annihilated by an order 3 operator. With $x = w - 1/2$, the unique singular series

$$x^{7/2} \cdot (1 - 41x/6 + 26\,557x^2/792 - 8692\,015x^3/61\,776 + O(x^4)) \qquad (7)$$

yields an order 6 right division operator L_6. By proceeding through a sequence of such solutions, one can eliminate much if not all of the p-fold searching described in [9] to achieve the factorization $L_{29} = L_5 \cdot L_{24} = L_5 \cdot L_{12}^{(\mathrm{left})} \cdot L_{12}^{(\mathrm{right})}$ and the additional factorization of $L_{12}^{(\mathrm{right})}$.

The points chosen for series expansion need not be restricted to the rational head polynomial roots as in equations (6) and (7) above. The factor $1 + 3w + 4w^2$, which appeared already in the head polynomial of the L_7 that annihilated $\tilde{\chi}^{(3)}$, has 'accidental' modulo prime factorizations for roughly half the primes close to 2^{15}. For example, with prime $p = 32\,719$, one has $1 + 3w + 4w^2 = 4(w - 8973)(w - 31\,925)$ modulo p. We can write $x = w - w_p$ with w_p either 8973 or 31 925 and obtain solutions about $x = 0$ satisfying $L_{24}(S(x) \cdot \ln^2(x) + R(x) \cdot \ln(x) + Q(x)) = 0$ modulo p, where R and Q are regular at $x = 0$ and the series $S(x) = x + O(x^2)$ is unique. Testing shows that $S(x)$ is annihilated (modulo p) by an order 3 operator $L_3(x)$. As we show in general in appendix C, one can obtain from $L_3(x)$ a right division (modulo p) operator $L_3(w)$ and then from multiple modulo prime calculations, an exact right division $L_3(w)$ by the Chinese remainder theorem. This reconstructed $L_3(w)$[16] has the factor $1 + 3w + 4w^2$ in its head polynomial in spite of the fact that w_p is clearly not one of the roots $(-3 \pm \mathrm{i}\sqrt{7})/8$ modulo p of $1 + 3w + 4w^2$.

Testing at points other than $w = 0$ also enables one to exclude certain series solutions as generators of right division operators. For example, the case 3 polynomial[17] $1-7w+5w^2-4w^3$ which is a factor of the head polynomial of L_{24} has the modulo prime, $p_0 = 32\,749$, factorization $32\,745 \cdot (w^2 + 11\,821w + 10\,836)(w - 3635)$. Define $x = w - 3635$. Then the singular solution modulo p_0 about $x = 0$ is $S(x)\ln(x) + R(x)$ with

$$S(x) = x^5 + 13\,877x^6 + 9339x^7 + 25\,021x^8 + 21\,884x^9 + O(x^{10}) \qquad (8)$$

unique and $R(x)$ regular at $x = 0$. Testing the series (8) shows there is no $L_n(S) = 0$ modulo p_0 for any $n < 24$. Thus there is no operator of order less than 24 that has $x = w - 3635$ as a factor (modulo p_0) of the head polynomial and more generally, $1 - 7w + 5w^2 - 4w^3$ as a factor. This also implies that any solution S that is singular at some root of $1-7w+5w^2-4w^3$ and satisfying $L_{12}^{(\mathrm{left})}(S) = 0$ cannot also be a solution of an operator of order less than 12 that right divides $L_{12}^{(\mathrm{left})}$. The same conclusion is reached for the remaining case 3 and both case 4 polynomials for $\tilde{\chi}^{(5)}$ from appendix C in [8].

If we only knew L_{24} or $L_{12}^{(\mathrm{left})}$ modulo prime for a few primes, the above information about singular solutions associated with case 3 and 4 polynomials would not be particularly useful for finding or excluding factorization. However, with the exact L_{24} available, one has global

[15] More precisely the Maple 13 command DFactor(*, onestep) was not able to yield a conclusive answer for such complicated large order operators. Seeking for a left division of $L_{12}^{(\mathrm{left})}$ we had similar inconclusive Maple calculations on adj($L_{12}^{(\mathrm{left})}$), the adjoint operator of $L_{12}^{(\mathrm{left})}$.

[16] It is equivalent to the product $Z_2 \cdot N_1$ which was shown on general grounds in [9] to right divide L_{29}.

[17] We follow the notation of appendix C in [8].

information and can match series solutions about $w = 0$ to solutions about other singular points of L_{24}. In particular, if one can show that every series solution $S(w)$ about $w = 0$ satisfying[18] $L_{12}^{(\mathrm{left})}(S) = 0$ is singular at some root of the case 3 or 4 polynomials, then one has proved that $L_{12}^{(\mathrm{left})}$ does not factorize. Our demonstration that this is the case uses the singular point $w_s = 0.158\,53\ldots$ which is a root of $1 - 7w + 5w^2 - 4w^3$. This is a particularly convenient point as it is the closest root of the head polynomial of L_{24} to both $w = 0$ and $w = 1/4$.

We begin the demonstration by studying the two linearly independent solutions of the form $S_i = A_i(w) \cdot \ln^3(w)$ plus terms with lower powers of $\ln(w)$. The two series,

$$A_1 = 9w + 261w^3 + 1845w^4 + 7046w^5 + 42\,771w^6 + 145\,980w^7 + 785\,528w^8 + 2536\,628w^9 + 12\,800\,309w^{10} + 38\,627\,228w^{11} + 187\,738\,058w^{12} + \cdots + \alpha_{1,n} \cdot w^n + \cdots, \tag{9}$$

$$A_2 = 27w^2 + 102w^3 + 270w^4 + 2164w^5 + 5532w^6 + 43\,722w^7 + 132\,130w^8 + 922\,108w^9 + 3158\,590w^{10} + 19\,690\,882w^{11} + 72\,977\,164w^{12} + \cdots + \alpha_{2,n} \cdot w^n + \cdots, \tag{10}$$

satisfy $L_{24}(A_i) = 0$ *but not* $L_{12}^{(\mathrm{right})}(A_i) = 0$ and a ratio test shows they have radii of convergence $|w| = w_s = 0.158\,53\ldots$. Thus both A_1 and A_2 are singular at $w = w_s$ and cannot be generators of an L_n, $n < 24$, that right divides L_{24}. Whether a linear combination of A_1 and A_2 leads to a right division operator is now determined as follows.

Near $x = 0$ where $x = w_s - w$, the A_i must be of the form $A_i = B_i \cdot f(x) \cdot \ln(x) + g_i(x)$ where $f(x)$ and the $g_i(x)$ are all regular at $x = 0$. The series $A(w) \propto B_2 A_1(w) - B_1 A_2(w)$ will be regular at $w = w_s$ and if the amplitude ratio B_1/B_2 is rational, then $A(w)$ might be a candidate generator for a right division operator. In principle, the amplitudes B_i could be found by matching series solutions about $x = 0$ to those about $w = 0$ but since we only need the B_1/B_2 ratio, a simpler procedure that utilizes only the coefficients in (9) and (10) is possible. We note that of the remaining singularities of A_i, the nearest to $w = 0$ are at $|w| = 1/4$. This implies that $B_1/B_2 = \alpha_{1,n}/\alpha_{2,n}$, a ratio of coefficients from (9) and (10), to an accuracy of order $(4w_s)^n \approx 0.634^n$. Thus, the problem reduces to searching for a (small) rational B_1/B_2 from a sequence that converges exponentially. This can be done by expressing $\alpha_{1,n}/\alpha_{2,n}$ as a continued fraction. We observe that, for n greater than some fixed n_0, a particular term in the continued fraction grows exponentially which is a clear indication that in the limit $n \to \infty$ the continued fraction terminates and is the (small) rational $B_1/B_2 = -637/228$. Our result for the linear combination series is then

$$A(w) = 2052w + 17\,199w^2 + 124\,482w^3 + 592\,650w^4 + 2984\,956w^5 + O(w^6) \tag{11}$$

which is confirmed to have a radius of convergence $|w| = 1/4$. However, testing the series (11) shows it is not annihilated by any L_n, $n < 24$, and thus we have eliminated the only possible linear combination candidate series for a right division operator.

While direct testing is an easy way to exclude $A(w)$ in (11), such direct testing is impractical[19] as a general method for excluding the many possible series that arise in the remaining part of our proof. Instead we supplement direct testing by a method that relies on the global information provided by the exact L_{24}. As an illustration of this method, consider

[18] Equivalently every solution of L_{24} that is not a solution of $L_{12}^{(\mathrm{right})}$.

[19] This was the problem encountered in [9].

again $A(w)$. If one matches (11) to series about $w = 1/4$ one finds that $A(w)$ contains, as the leading logarithmic function, $A_s \cdot \ln^2(y)$ where $2y = 1 - 4w$ and

$$A_s = 1 - 21\,469\, y/640 - 1489\,293 y^2/81\,920 + 229\,328\,363 y^3/10\,485\,760 + O(y^4). \qquad (12)$$

A ratio test on the coefficients in (12) shows that A_s has a radius of convergence $|y| = 1/2 - 2w_s$ and thus is singular at $w = w_s$. This in turn implies that there are branches of the function $A(w)$ on which it is singular[20] at $w = w_s$. By forming the linear combination $A(w) \propto B_2 A_1(w) - B_1 A_2(w)$, we only succeeded in forcing an 'accidental' cancellation of the singularity at $w = w_s$ on the principal branch of the function. The singularity remains on at least some other branches and thus A is excluded as a generator of a right division operator for exactly the same reason as A_1 and A_2.

The above argument has excluded, as generators of right division operators, 8 of 12 linearly independent solutions satisfying $L_{12}^{(\mathrm{left})}(S) = 0$. These we take to be $C_1 \propto L_{12}^{(\mathrm{right})}(A)$ with A from (11) and $C_2 \propto L_{12}^{(\mathrm{right})}(A_2)$ with A_2 from (10) plus the six series with leading $\ln(w)$ dependence $C_i \cdot \ln^p(w)$, $p = 1, 2$ and 3. Explicitly,

$$\begin{aligned} C_1(w) = w^6 &- 444 w^7/11 + 275\,773\,109 w^8/129\,360 + 19\,252\,320\,091 w^9/194\,040 \\ &- 964\,738\,631\,897 w^{10}/388\,080 - 2082\,457\,681\,309 w^{11}/27\,720 \\ &+ 17\,517\,580\,633\,073\,581 w^{12}/17\,075\,520 + O(w^{13}), \end{aligned} \qquad (13)$$

$$\begin{aligned} C_2(w) = w^6 &+ 403\,206 w^7/1661 - 13\,446\,782\,071 w^8/19\,533\,360 \\ &- 2413\,114\,741\,889 w^9/29\,300\,040 - 4359\,267\,083\,039 w^{10}/1065\,456 \\ &- 875\,856\,906\,689\,449 w^{11}/4185\,720 \\ &+ 23\,619\,065\,101\,886\,078\,533 w^{12}/2578\,403\,520 + O(w^{13}). \end{aligned} \qquad (14)$$

Because $L_{12}^{(\mathrm{right})}$ does not have the factor $w - w_s$ in its head polynomial, the $C(w)$ functions carry the same $w = w_s$ singularities as the $A(w)$ from which they have been generated. Thus even though the series $C_1(w)$ has the radius of convergence $|w| = 1/4$, the analytically continued function $C_1(w)$ is still singular at $w = w_s$ on some other branches. The series $C_2(w)$ has the radius of convergence $|w| = w_s$ and is already singular at $w = w_s$ on the principal branch.

It was shown in [9] that the remaining four solutions satisfying $L_{12}^{(\mathrm{left})}(S) = 0$ are of the form $C_i(w) \cdot \ln(w) + D_i(w)$ and $C_i(w)$, $i = 3, 4$, with the C_i and D_i regular at $w = 0$. These two C_i, together with the two in (13) and (14), are linearly independent and can all be generated from the coefficient of $\ln(w)$ in $L_{12}^{(\mathrm{right})}(S_{24})$ where S_{24} contains four arbitrary constants and is of the form $S_{24} = F(w) \cdot \ln^2(w) + G(w) \cdot \ln(w) + H(w)$ with F, G and H all regular at $w = 0$. We demand that F satisfies[21] $L_{12}^{(\mathrm{right})}(F) = 0$. This guarantees that $C(w) \cdot \ln(w)$ is the leading logarithm in $L_{12}^{(\mathrm{right})}(S_{24})$ and simplifies the subsequent analysis. Only F and G are relevant for determining C and a possible choice is

$$\begin{aligned} F = \beta_1 \cdot (3177 w &- 174\,840 w^4 - 817\,828 w^5 - 5829\,558 w^6 \\ &- 25\,983\,762 w^7 - 142\,882\,882 w^8 - 620\,769\,318 w^9 - 3086\,072\,424 w^{10} \end{aligned}$$

[20] These singularities can be reached, for example, by following a path along the real axis from $w = 0$ to w just less than 1/4, circling $w = 1/4$ any number N of times, and then moving back to $w = w_s$ along the real axis. Since A_s multiplies the leading (unique) logarithmic singularity at $w = 1/4$, there cannot be cancellation of the $w = w_s$ singularity of A_s on all branches distinguished by N.

[21] For the explicit F in (15), it happens that $\mathrm{d}F/\mathrm{d}\beta_1$ is annihilated by an L_{11} and $\mathrm{d}F/\mathrm{d}\beta_2$ by an L_9. This reduction from L_{12} plays no role in our subsequent arguments.

$$- 13\,199\,839\,762w^{11} - 62\,214\,586\,728w^{12} + O(w^{13}))$$
$$+ \beta_2 \cdot (3177w^3 + 13\,803w^4 + 74\,932w^5 + 287\,997w^6$$
$$+ 1265\,280w^7 + 4296\,418w^8 + 17\,162\,736w^9 + 48\,945\,231w^{10}$$
$$+ 173\,557\,768w^{11} + 284\,486\,847w^{12} + O(w^{13})) \tag{15}$$

and

$$G = \beta_1 \cdot (1604\,883\,673w^8/210 + 2823\,208\,099w^9/105$$
$$+ 47\,115\,755\,881w^{10}/140 + 782\,148\,892\,459w^{11}/630 + O(w^{13}))$$
$$- \beta_2 \cdot (366\,106\,439w^8/1050 + 1576\,821\,038w^9/525$$
$$+ 9206\,778\,909w^{10}/350 + 1049\,578\,781\,449w^{11}/6300 + O(w^{13}))$$
$$+ \beta_3 \cdot (35w^6 + 1223w^8 + 1852w^9 + 36\,064w^{10} + 96\,388w^{11} + O(w^{13}))$$
$$+ \beta_4 \cdot (105w^7 + 304w^8 + 3536w^9 + 10\,192w^{10} + 79\,089w^{11} + O(w^{13})), \tag{16}$$

where the β_i are arbitrary constants. We will now show that no choice of these constants can yield a $C = C(\beta_1, \beta_2, \beta_3, \beta_4)$, defined by

$$L_{12}^{(\text{right})}(F \cdot \ln^2(w) + G \cdot \ln(w) + H) = C \cdot \ln(w) + D, \tag{17}$$

that is a generator for a right division operator of $L_{12}^{(\text{left})}$. The argument is essentially that given above for the exclusion of C_1 and C_2 in (13) and (14). In fact the demonstration has already been partially completed since, in terms of $C(\beta_1, \beta_2, \beta_3, \beta_4)$, $C_1 \propto C(0, 0, 3, 4)$ and $C_2 \propto C(0, 0, 453, 1744)$. Since F in (15) satisfies $L_{12}^{(\text{right})}(F) = 0$ it is not singular at $w = w_s = 0.158\,53\ldots$, neither is $L_{12}^{(\text{right})}(F \cdot \ln^2(w))$. Thus it suffices to investigate G and if every G is singular at $w = w_s$ then so is C defined by (17) and we have proved that $L_{12}^{(\text{left})}$ does not factorize.

A ratio test on the series coefficients in (16) shows the generic G has a radius of convergence $|w| = w_s$ and thus is singular at $w = w_s$. But by the same analysis that led from the series (9) and (10) to the linear combination (11), we can construct three $G = G(\beta_1, \beta_2, \beta_3, \beta_4)$ each of whose radius of convergence is $|w| = 1/4$. These are

$$G(105, 0, 0, -1182\,781), \qquad G(0, 525, 0, -1443\,727), \qquad G(0, 0, 3, 4). \tag{18}$$

To the remaining linearly independent $G = G(0, 0, 0, 1)$, one can add any combination of the three in (18) but this will not change its radius of convergence from $|w| = w_s$ and remove the singularity at $w = w_s$. In this sense, $G(0, 0, 0, 1)$ is equivalent to $G(0, 0, 453, 1744)$ which corresponds to C_2 via (17) and is excluded as a generator of any right division operator.

To determine the behaviour of the three G functions in (18) in the vicinity of $y = 0$ where $2y = 1 - 4w$ we match the series[22] $2F \cdot \ln(w) + G$ in w about $w = 0$ to solutions S satisfying $L_{24}(S) = 0$ about $y = 0$. Since $L_{12}^{(\text{right})}(F) = 0$ one can show that F (and $F \cdot \ln(w)$) can contain only the first power of $\ln(y)$ near $y = 0$. Any $\ln^2(y)$ or $\ln^3(y)$ we find in the matching S can only come from G in the combination solution $2F \cdot \ln(w) + G$. Our matching shows that the leading logarithmic dependences of the three $G's$ in (18) are respectively

$$(-3420\,025/8192/\pi^2)A_s \ln^3(y), \qquad (-2473\,625/16\,384/\pi^2)A_s \ln^3(y),$$
$$(-125/16\,384/\pi^2)A_s \ln^2(y), \tag{19}$$

where A_s is given by (12). A linear combination of the first two $G's$ in (18) can be constructed to eliminate the leading $A_s \ln^3(y)$ shown in (19) and we find that the

[22] It follows from $L_{24}(F \cdot \ln^2(w) + G \cdot \ln(w) + H) = 0$ by analytic continuation around the $w = 0$ singularity that also $L_{24}(2F \cdot \ln(w) + G) = 0$.

resultant $G(494\,725, -6840\,050, 0, 13\,236\,968\,029)$ has $A_s \cdot \ln^2(y)$ as the leading logarithmic singularity. Clearly this remaining singularity can now be eliminated by forming a linear combination with the last G in[23] (18). In summary, we have generated the three $G(\beta_1, \beta_2, \beta_3, \beta_4)$ combinations

$$G(494\,725, -6840\,050, 15\,276\,842\,775, 33\,606\,091\,729),$$
$$G(0, 525, 0, -1443\,727), \qquad G(0, 0, 3, 4) \tag{20}$$

which are irreducible in the sense that no further superposition can eliminate the $A_s \cdot \ln^3(y)$ and $A_s \cdot \ln^2(y)$ singularities of the last two $G's$ while the first G is unique in that it contains neither $A_s \cdot \ln^3(y)$ nor $A_s \cdot \ln^2(y)$. The last G in (20) has already been identified as being associated with C_1. We associate the middle G in (20) with $C_3 \propto C(0, 525, 0, -1443\,727)$ which cannot be a generator of a right division operator of $L_{12}^{(\mathrm{left})}$ for exactly the same reason as C_1. We associate the first G in (20) with $C_4 \propto C(494\,725, -6840\,050, 15\,276\,842\,775, 33\,606\,091\,729)$ and test C_4 directly. We find there is no operator satisfying $L_n(C_4) = 0$ with $n < 12$ and this completes our proof that $L_{12}^{(\mathrm{left})}$ does not factorize. The explicit new C_i supplementing those in (13) and (14) are

$$\begin{aligned} C_3(w) = w^6 &+ 281\,575\,923w^7/34\,167\,793 + 48\,755\,202\,697\,119w^8/8371\,109\,285 \\ &+ 788\,146\,152\,364\,265w^9/5022\,665\,571 \\ &- 48\,321\,460\,210\,711\,729w^{10}/33\,484\,437\,140 \\ &- 1046\,678\,480\,403\,963\,299w^{11}/4783\,491\,020 \\ &- 4705\,373\,665\,277\,858\,926\,411w^{12}/3683\,288\,085\,400 + O(w^{13}), \end{aligned} \tag{21}$$

$$\begin{aligned} C_4(w) = w^6 &- 124\,536w^7/649 + 2840\,488\,261w^8/508\,816 + 39\,013\,193\,251w^9/254\,408 \\ &- 24\,532\,098\,411\,899w^{10}/9540\,300 - 6082\,145\,734\,733w^{11}/68\,145 \\ &- 179\,169\,570\,633\,725\,593w^{12}/314\,829\,900 + O(w^{13}). \end{aligned} \tag{22}$$

3.1. *More on the structure of the differential operator* $L_{12}^{(\mathrm{left})}$

Once the differential operator $L_{12}^{(\mathrm{left})}$ has been proved to be irreducible, one may wonder whether this high-order differential operator can nevertheless be built from factors of lower order. A high-order differential operator can be irreducible and still result from 'operations' involving differential operators of lower order since it may be a symmetric power of a lower order differential operator or a symmetric product of two (or more) lower order differential operators.

The symmetric nth power of a differential operator L_q is the differential operator whose corresponding ODE annihilates a generic linear combination of the q solutions of L_q to the power n. The symmetric nth power of a differential operator L_q of order q has order $\frac{(q+n-1)!}{(q-1)!n!}$.

The symmetric product of the differential operators L_{q_1} and L_{q_2} of orders q_1 and q_2, respectively, is the differential operator whose ODE annihilates the product of a generic linear combination of the q_1 solutions of L_{q_1} and a generic linear combination of the q_2 solutions of L_{q_2}. This symmetric product is of order[24] $q_1 \cdot q_2$.

[23] The surprise, at least for us, is that all of the necessary combinations can be formed with rational amplitudes.

[24] The order $q = q_1 \cdot q_2$ is for the generic case. In general the order of the symmetric product is $q_1 + q_2 - 1 \leqslant q \leqslant q_1 \cdot q_2$.

We use the notation $[w^p]$ to indicate a series that starts as w^p (const + $\cdots$). In [9] it was shown that the formal solutions of $L_{12}^{(\text{left})}$ at $w = 0$ follow this scheme. There are two sets of four solutions ($k = 6,\ 7$):

$$\begin{aligned}
&[w^k]\ln(w)^3 + [w^5]\ln(w)^2 + [w]\ln(w) + [w],\\
&[w^k]\ln(w)^2 + [w^5]\ln(w) + [w],\\
&[w^k]\ln(w) + [w], \qquad \text{and} \qquad [w^k]
\end{aligned} \tag{23}$$

and two sets of two solutions ($k = 8,\ 9$)

$$[w^k]\ln(w) + [w] \qquad \text{and} \qquad [w^k], \tag{24}$$

$$[w^k]\ln(w) + [w] \qquad \text{and} \qquad [w^k]. \tag{25}$$

We denote by BLn a set of solutions such as (23) containing $n+1$ solutions with a logarithmic solution of maximal degree n. For the scheme above, we thus have two $BL3$ blocks and two $BL1$ blocks, and in each block there is also a non-logarithmic solution.

We first consider the possibility that $L_{12}^{(\text{left})}$ is a symmetric power of an operator of lower order. It is straightforward to see that the only possibility is that $L_{12}^{(\text{left})}$ could be a symmetric 11th power of a differential operator of order 2 with one $BL1$ block. This possibility is ruled out, since there is no $BL11$ block in the solutions of $L_{12}^{(\text{left})}$.

Next, for the possibility that $L_{12}^{(\text{left})}$ is a symmetric product of differential operators of lower order, we consider only the cases where the product has the maximal order. There are three cases to consider. The symmetric product of differential operators of orders 2 and 6 (configuration denoted $2\cdot 6$), 3 and 4 (configuration $3\cdot 4$), or 2, 2 and 3 (configuration $2\cdot 2\cdot 3$).

The symmetric product of two differential operators L_1 and L_2 containing the blocks BLn_1 and BLn_2, respectively, should contain in its solutions the block BLn with $n = n_1 + n_2$. Since the differential operator $L_{12}^{(\text{left})}$ contains two $BL3$ blocks, the case $2\cdot 2\cdot 3$ is ruled out.

Let us detail the compatibility of the case $3\cdot 4$ at $w = 0$. With two $BL3$ blocks in $L_{12}^{(\text{left})}$ the only possibility is that the order 3 differential must have one $BL2$ block:

$$\begin{aligned}
&S_1 \ln(w)^2 + S_{11}\ln(w) + S_{10},\\
&S_1 \ln(w) + S_{20},\\
&S_1
\end{aligned} \tag{26}$$

and the order 4 differential operator must have two $BL1$ blocks:

$$\begin{aligned}
&T_1 \ln(w) + T_{10},\\
&T_1,
\end{aligned} \tag{27}$$

$$\begin{aligned}
&V_1 \ln(w) + V_{10},\\
&V_1.
\end{aligned} \tag{28}$$

It is a simple calculation to form the product of a combination from the set $BL2$ with a combination of the solutions from the two sets $BL1$. One obtains

$$\begin{aligned}
&S_1\cdot T_1\ln(w)^3 + (S_1\cdot T_{10} + S_{11}\cdot T_1)\ln(w)^2 + (S_{10}\cdot T_1 + S_{11}\cdot T_{10})\ln(w) + S_{10}\cdot T_{10},\\
&S_1\cdot T_1\ln(w)^2 + S_{11}\cdot T_1\ln(w) + S_{10}\cdot T_1,
\end{aligned} \tag{29}$$

$$S_1\cdot T_1\ln(w) + S_{20}\cdot T_1, \tag{30}$$

$$S_1\cdot T_1 \tag{31}$$

and a similar set of four solutions with V's instead of T's. These eight solutions correspond to the two $BL3$ occurring for $L_{12}^{(\mathrm{left})}$ at $w = 0$. One also obtains

$$S_1 \cdot T_1 \ln(w)^2 + (S_1 \cdot T_{10} + T_1 \cdot S_{20}) \ln(w) + S_{20} \cdot T_{10}, \tag{32}$$

$$S_1 \cdot T_1 \ln(w) + S_1 \cdot T_{10} \tag{33}$$

and two other solutions where V's replace T's. Subtracting (32) from (29) and (33) from (30), one obtains

$$(S_1 \cdot T_{10} + T_1 \cdot S_{20} - T_1 \cdot S_{11}) \ln(w) + (S_{20} \cdot T_{10} - S_{10} \cdot T_1), \tag{34}$$

$$S_1 \cdot T_{10} - T_1 \cdot S_{20}. \tag{35}$$

Note that in a set of solutions such as $BL2$ above, the series S_{11} depends on S_{20} and S_1 and can be expressed as

$$S_{11} = \alpha S_1 + 2S_{20}. \tag{36}$$

The coefficient 2 is generic for any order 3 ODE and α is a constant that depends on the ODE at hand. Inserting this S_{11} in (34), we can arrange to have the non-logarithmic series the same as the series in front of the log. This is then one of the $BL1$ blocks occurring in $L_{12}^{(\mathrm{left})}$. The second $BL1$ block is obtained by considering (34) with V's instead of T's.

We have thus shown that under the hypothesis that $L_{12}^{(\mathrm{left})}$ is a symmetric product of two factors, the scheme of solutions at $w = 0$ is compatible with the configuration $3 \cdot 4$. Similar calculations show that the configuration $2 \cdot 6$ is also compatible and in this case the order 2 operator has a $BL1$ block and the order 6 operator has two $BL2$ blocks.

Next we must check for each configuration ($3 \cdot 4$ and $2 \cdot 6$) whether or not the symmetric product is compatible with the scheme of solutions for $L_{12}^{(\mathrm{left})}$ at those other singularities containing enough logarithmic solutions. We recall that the scheme at $w = \infty$ is the same as the scheme at $w = 0$. The scheme of solutions of $L_{12}^{(\mathrm{left})}$ at the singularity $w = 1/4$ is one $BL3$ and four $BL1$ blocks. One sees immediately that the configuration $3 \cdot 4$ is ruled out. For any set of solutions that we attach to the order 3 and order 4 differential operators, we end up with either more than one $BL3$ or at least one $BL2$. The configuration $2 \cdot 6$ is acceptable, since in this case there can be one $BL1$ block in the order 2 operator and one $BL2$ plus three $BL0$ blocks in the order 6 operator. It remains to be seen whether or not the configuration $2 \cdot 6$ is compatible with the scheme of solutions at the point $w = -1/4$, which is two $BL2$, one $BL1$ and four $BL0$ blocks. Our checks show that the configuration $2 \cdot 6$ is ruled out.

In conclusion we have shown that the differential operator $L_{12}^{(\mathrm{left})}$ is not a symmetric n'th power of a lower order operator nor is it a symmetric product of two (or more) operators of orders q_1 and q_2 under the hypothesis that the order of the symmetric product reaches its maximum value $q_1 \cdot q_2 = 12$. It should be noted that our considerations regarding symmetric powers/products (unlike the question about the factorization of $L_{12}^{(\mathrm{left})}$) do not require knowledge about $L_{12}^{(\mathrm{left})}$ in exact arithmetic. The block structure of the solutions can be obtained from the operator modulo a prime.

4. Analytic continuation of $\Phi^{(5)}$ and its behaviour at $w = 1/2$

Our analytic continuation of $\Phi^{(5)}$ is limited to paths that follow the real w axis on the intervals $[0,\ 1/4]$, then $[1/4,\ (3-\sqrt{5})/2]$ and finally $[(3-\sqrt{5})/2),\ 1/2]$. We allow any number of half-integer turns about the ferromagnetic point $w = 1/4$, that is, rotation by any angle $\theta = n\pi$

with n odd. This is followed by any rotation $\theta = m\pi$, m odd, around $w = (3-\sqrt{5})/2$. This point is one of the s-plane circle singularities discussed by Nickel [4, 5]. The point $w = 1/2$ also maps onto the $|s| = 1$ circle but it is not a singularity of $\tilde{\chi}^{(5)}$ when it is approached on any path that does not leave the principal disc $|s| \leqslant 1$. In terms of our n, m paths in the w-plane, the combinations $n = m = \pm 1$ are such principal disc constrained paths. Every other n, m combination is a path that reaches $w = 1/2$ on another branch.

Our starting point for the analytic continuation is the $\Phi^{(5)}$ series expansion about $w = 1/4$ in appendix B. On the physical interval $0 \leqslant w < 1/4$, $2y = 1 - 4w$ is positive real and the half-integer turns about $w = 1/4$ that bring one to $w > 1/4$ simply requires the replacements

$$y \longrightarrow -y, \qquad \ln(y/4) \longrightarrow \ln(|y|/4) - \mathrm{i}n\pi, \tag{37}$$

in (B.3) with n understood to be odd. The new (B.3) series generated with the replacements (37) is now to be matched to series in z where $z = 3-\sqrt{5}-2w = 5/2-\sqrt{5}-y$. Although direct matching is possible, considerable improvement in the utility of the y series results by first making an Euler transformation by the replacement $y \to y/(1-y)$. This has the effect of bringing the $w = (3-\sqrt{5})/2 \approx 0.382$ singular point closer to $w = 1/4$ while moving $w_s \approx 0.1585$ further away.

To generate $\Phi^{(5)}$ series in z we must first analytically continue the elliptic integrals in (4). The replacements required for $w > 1/4$, with the same n as in (37), are

$$\begin{aligned} K(4w) &\longrightarrow u[K(u) + \mathrm{i}nK(u')], \\ E(4w) - K(4w) &\longrightarrow [E(u) - K(u) - \mathrm{i}nE(u')]/u, \end{aligned} \tag{38}$$

where $u = 1/(4w)$ and $u' = \sqrt{1-u^2}$. The new elliptic integrals (38) are easily developed as series in z from their defining differential equations set up as recursion relations. The remaining step of finding a particular integral of (2) together with all homogeneous series solutions is also straightforward. The ODE numerical recursion in z is not as unstable as the recursion in y noted in appendix B. Here one loses only about a factor 10 in relative accuracy for each two orders in z.

The matched series in z is of the form $A(z) \cdot \ln(z) + B(z)$ where A and B are regular at $z = 0$. Since $\tilde{\chi}^{(1)}$ and $\tilde{\chi}^{(3)}$ are not singular at this point, we can identify the singularity in $\Phi^{(5)}$ with that in $\tilde{\chi}^{(5)}$. We find that the leading singular term is[25]

$$[(1+10n^2+5n^4)/16] \cdot [(25/693)(5-\sqrt{5})(2+\sqrt{5})^{11}/(2^{22}\pi^2)] \cdot z^{11}\ln(z). \tag{39}$$

The $n = 1$ singularity, when mapped to s-plane variables, agrees with the sum contribution $\tilde{\chi}^{(5)}_{0,1} + \tilde{\chi}^{(5)}_{0,-1}$ from equation (14) in [4]. The new result in (39) is the branch-dependent multiplicity $(1+10n^2+5n^4)/16$.

To most clearly identify a possible $w = 1/2$ singularity in the $A(z)$ and $B(z)$ series, we make another Euler transformation with the replacement $z \to z/(1+6z/5)$. This moves the known singularity at $w = 1/4$ to the new $z = 5(\sqrt{5}-1)/16 \approx 0.386$ and the potential singularity at $w = 1/2$ to the new $z = -5(16-5\sqrt{5})/131 \approx -0.184$. There is another potential singularity at a complex root of the case 4 polynomial $1-w-3w^2+4w^3$. This maps to the new $|z| \approx 0.387$. Since the singularity of interest is about factor 2.1 closer than the next nearest, any singularity at $w = 1/2$ will be observable in an N term Euler transformed z series with corrections of order $(2.1)^{-N}$. This factor $(2.1)^{-N}$ is also the bound we can put on any $w = 1/2$ singularity amplitude if a ratio test of coefficients does not indicate a singularity at $z \approx -0.184$. We find the absence of such a singularity for the $A(z)$ series which we have generated to length $N = 1100$. Because the $A(z)$ series multiplies $\ln(z)$, this also indicates

[25] This is based on floating point results to about 300 digit accuracy.

that any possible singularity at $w = 1/2$ will have an amplitude independent of the index m specifying the logarithmic branch of the $z = 0$ ($w = (3 - \sqrt{5})/2$) singularity.

We find that for $n = \pm 1$, the $B(z)$ series is also not singular at $z \approx -0.184$. For other n values, a singularity is clearly indicated and by a coefficient ratio analysis of different n series completely analogous to what was done for the A_1 and A_2 series in (9) and (10), we find that the singularities at $w = 1/2$ have amplitudes proportional to the branch-dependent multiplicity factor $(n^2 - 1)^2$. A more detailed analysis involving explicit fitting of the $B(z)$ series coefficients yields the singularity amplitude which we have confirmed by direct matching of the $\tilde{\chi}^{(5)}$ series in z to ODE solution series about $w = 1/2$. The result for the singular part of $\tilde{\chi}^{(5)}$ at $w = 1/2$ is

$$\tilde{\chi}^{(5)}_{\rm sing} = [(n^2 - 1)^2 (2\sqrt{6})/(315\pi)] \cdot (1 - 2w)^{7/2} \times [1 + 41(1 - 2w)/12 + 26\,557(1 - 2w)^2/3168 + \cdots], \qquad (40)$$

where the amplitude has been verified to our numerical accuracy of about 250 digits. We have not identified a Landau singularity associated with (40). Finding this Landau singularity in the $\tilde{\chi}^{(5)}$ integrand remains as the major unsolved challenge of this paper.

5. Conclusion

We have completed the quest begun in [8] for the exact integer arithmetic ODE satisfied by $\tilde{\chi}^{(5)}$. While most explicit results are far too extensive to be published here, a selection can be found on the web site [10]. These include L_{24} and $E^{(5)}$ defined in (2)–(4), the explicit factorization $L_{24} = L_{12}^{(\rm left)} \cdot L_{12}^{(\rm right)}$, the operator L_{29} and the exact integer high temperature series for $\tilde{\chi}^{(5)}$ to 8000 terms generated from (2)–(4). Also included are numerical coefficients to 800 digits for the $\tilde{\chi}^{(5)}$ series at the ferromagnetic point to supplement (B.4).

We have used the exact ODE to resolve at least one issue that was left undecided in [9]. In particular, we have shown in section 3 that the operator $L_{12}^{(\rm left)}$ cannot be factored. The techniques we have described, both for factorization and proving the converse, are not all well known and we expect they will find application in other problems. In addition we have shown that $L_{12}^{(\rm left)}$ is not a symmetric n'th power of a lower order operator nor is it a symmetric product of two (or more) operators if the order of the symmetric product reaches its maximum value.

An issue raised in [8] has only been partially resolved. No obvious candidate for a Landau singularity in the integral representation of $\tilde{\chi}^{(5)}$ at $w = 1/2$ could be found and because a Landau integrand singularity is a necessary condition[26] [7] for an integral to be singular, it was conjectured that $\tilde{\chi}^{(5)}$, necessarily defined by analytic continuation from its series representation, would not be singular at $w = 1/2$[27]. We have now, in section 4, shown that $\tilde{\chi}^{(5)}$ is singular at $w = 1/2$ and so clearly have identified all the singularities of $\tilde{\chi}^{(5)}$ by an analysis of the ODE it satisfies. The $w = 1/2$ singularity in $\tilde{\chi}^{(5)}$ proves the existence of an associated Landau integrand singularity, but we have made no progress in identifying this Landau singularity. And so our goal of unifying the Landau integrand analysis with the ODE approach to the behaviour of the general $\tilde{\chi}^{(n)}$ continues to elude us.

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[26] Hwa and Teplitz (see page 6 in [7]) give [14, 15] as the original sources for the necessary requirement.
[27] The abstract in [8] erroneously stated the absence of a singularity without caveat. In view of the findings of this paper the conclusion that $\tilde{\chi}^{(6)}$ is singularity free at $w^2 = 1/8$ must now also be discounted.

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Appendix A. The exact arithmetic $L_{12}^{(\mathrm{right})} = L_1 \cdot L_{11}$

It was reported in [9] that $L_{12}^{(\mathrm{right})} = L_1 \cdot L_{11}$ with a solution S_{12} that satisfies $L_{11}(S_{12}) = P$, $L_1(P) = 0$ where P is a polynomial. The order 11 linear differential operator L_{11} has a direct-sum decomposition

$$L_{11} = (Z_2 \cdot N_1) \oplus V_2 \oplus \left(F_3 \cdot F_2 \cdot L_1^s\right), \qquad (A.1)$$

where Z_2 is a second-order operator also occurring in the factorization of the linear differential operator associated with $\tilde{\chi}^{(3)}$, V_2 is a second-order operator equivalent to the second-order operator associated with $\tilde{\chi}^{(2)}$ and F_3 and F_2 are known in exact arithmetic [9]. The exact L_{11} was obtained in [9] from (A.1) and is given[28] in [10]. Our goal here is to determine the exact $L_{12}^{(\mathrm{right})}$ or equivalently the exact solution S_{12} which gives the polynomial $P = L_{11}(S_{12})$ and then L_1.

One possible form for this solution is

$$S_{12} = \ln(w) \cdot S_9 - 2w^3 - 6w^4 - 136w^5/3 - 3161w^6/24 - 3073w^7/4 - 245\,147w^8/120 - 334\,031w^9/30 - 872\,442w^{10}/35 - 19\,192\,553w^{11}/140 + O(w^{13}), \qquad (A.2)$$

where

$$S_9 = w^2 + 4w^3 + 30w^4 + 120w^5 + 690w^6 + 2760w^7 + 14\,280w^8 + 57\,120w^9 + 279\,090w^{10} + 1116\,360w^{11} + 5261\,244w^{12} + O(w^{13}) \qquad (A.3)$$

is annihilated by the order 9 operator $V_2 \oplus N_1 \oplus \left(F_3 \cdot F_2 \cdot L_1^s\right)$. The rational coefficients in (A.2) are sufficiently small that the four modulo prime solutions of (2) reported in [9] will yield S_{12} exactly to $O(w^{19})$. This in turn means that $P = L_{11}(S_{12})$ modulo any prime is given exactly to $O(w^{19})$. We now observe that the degree of P depends on the L_{11} representation and is 18 when either the minimum degree or minimum degree plus one representation of L_{11} is used. This implies that, with this degree restriction on L_{11} and without any further input from $\Phi^{(5)}$, the equation $P_{18} = L_{11}(S_{12})$ determines P_{18} modulo any prime exactly. The same equation can then be used in the recursion mode to extend the known S_{12} modulo a prime series of $O(w^{19})$ to any desired length. Clearly these steps can be repeated for as many primes as necessary to get the S_{12} coefficients in exact arithmetic by rational reconstruction. Then, once S_{12} has been extended to $O(w^{137})$, one can switch back to the minimum-order representation of L_{11} to get $P_{136} = L_{11}(S_{12})$, followed by L_1 and the minimum-order $L_{12}^{(\mathrm{right})}$ in exact arithmetic. Note that we make no attempt to reconstruct P_{18} in exact arithmetic. This polynomial is of no particular use and since it is generated from a non-minimal order L_{11} we expect its coefficients to be equally outrageously large.

The polynomial P_{136} is proportional to

$$5544w^5 + 197\,765w^6 + 419\,883\,469w^7/70 - 24\,564\,638\,026w^8/35 + \cdots - 2^{171} \cdot 134\,196\,985w^{135}/363 - 2^{173} \cdot 80\,400\,525w^{136}/847 \qquad (A.4)$$

and the operator L_1 that annihilates it is obviously proportional to $P_{136}\frac{\mathrm{d}}{\mathrm{d}w} - \frac{\mathrm{d}P_{136}}{\mathrm{d}w}$. The straightforward operator multiplication $L_1 \cdot L_{11}$ yields an $L_{12}^{(\mathrm{right})}$ and this can be normalized to

[28] Note that L_{11} in [10] is given in the monic form. Here L_{11} is understood to be normalized to a sum of products of polynomials times powers of $w \cdot \mathrm{d}/\mathrm{d}w$.

the monic or non-monic form as desired. Note that P_{136}/w^5 becomes the apparent singularity part of the head polynomial of $L_{12}^{(\rm right)}$. The corresponding apparent polynomial $P_{\rm app}$ of L_{11} divides out of the product $L_1 \cdot L_{11}$ as a common factor; this serves as a useful check of our algebra. The minimum-order $L_{12}^{(\rm right)}$ we have obtained can be found in [10].

The fact that the series S_9 multiplying the logarithm in (A.2) is annihilated by an operator of order $M = 9 < 11$ suggests that it might be possible to 'push' either V_2 or Z_2 to the left of L_1. And indeed we found that a U_1, equivalent to L_1, occurs at the left of the $L_9 = V_2 \oplus N_1 \oplus (F_3 \cdot F_2 \cdot L_1^s)$ that annihilates S_9 in (A.3). Next, our aim is to see whether U_1 can be pushed further to the right. This amounts to considering the various factorizations of L_9; for instance, the factorization where the equivalent of V_2 occurs at the leftmost position and there exists a direct sum of this with U_1. There is such a direct sum showing that an equivalent to U_1 occurs at the left of $N_1 \oplus (F_3 \cdot F_2 \cdot L_1^s)$. The process continues until it is no longer possible to find a direct sum of our equivalent to L_1 and any one of the right factors. This final factorization has a solution

$$\begin{aligned} S_5 = \ln(w) \cdot w^2 \cdot [1 + \sqrt{(1-4w)/(1+4w)}]/(1-4\,w)^2 + 64\,w^5/9 + 200\,w^6/9 + 712\,w^7/3 \\ + 10\,708\,w^8/15 + 253\,888\,w^9/45 + 1771\,852\,w^{10}/105 + 12\,245\,672 w^{11}/105 \\ + 331\,735\,612\,w^{12}/945 + O(w^{13}), \end{aligned} \qquad (A.5)$$

which can replace S_{12} from (A.2) as one of the 12 linearly independent solutions of $L_{12}^{(\rm right)}$. The operator that annihilates S_5 is

$$U_5 = \tilde{L}_1 \cdot \big(N_1 \oplus \big(F_2 \cdot L_1^s\big)\big), \qquad (A.6)$$

where if we define $U_4 = N_1 \oplus (F_2 \cdot L_1^s)$ to be given in our standard non-monic form, then $U_4(S_5) \propto P_{19} w^2/(1-4w)$ with

$$\begin{aligned} P_{19} = 9 + 36w + 18w^2 - 2064w^3 + 4581w^4 + 59\,584w^5 - 143\,476w^6 - 898\,464w^7 \\ + 124\,724w^8 + 813\,120w^9 + 9220\,240w^{10} + 55\,704\,896w^{11} + 65\,556\,224w^{12} \\ - 253\,883\,392w^{13} - 406\,194\,176w^{14} + 1318\,182\,912w^{15} + 2053\,013\,504w^{16} \\ + 368\,443\,392w^{17} - 454\,033\,408w^{18} - 272\,629\,760w^{19}. \end{aligned} \qquad (A.7)$$

The operator that annihilates $U_4(S_5)$ is

$$\tilde{L}_1 = [w(1-4w)P_{19}]\frac{\mathrm{d}}{\mathrm{d}w} - \left[(2-4w)P_{19} + w(1-4w)\frac{\mathrm{d}P_{19}}{\mathrm{d}w}\right] \qquad (A.8)$$

and again it is straightforward to combine this $\tilde{L}_1$ with the other exact arithmetic operators in (A.6) to get U_5.

An incidental remark on the solution (A.5) is that this is one of the few examples of a (partial) analytic solution of $L_{12}^{(\rm right)}$. Other simple solutions are $w^2/(1-4\,w)^2$, $w^2/(1-4\,w)/\sqrt{1-16\,w^2}$ and $w^2\,(1+4\,w)\,{}_2F_1(3/2,3/2;1;16\,w^2)$, $w^2\,(1+4\,w)\,{}_2F_1(3/2,3/2;3;1-16\,w^2)$ associated with L_1^s, N_1 and V_2, respectively. A more complicated case is that of Z_2 for which we refer[29] the reader to [9].

In conclusion, every solution of $L_{12}^{(\rm right)}$ can now be obtained as a linear combination of solutions of operators of order $M \leqslant 6$.

Appendix B. $\tilde{\chi}^{(5)}$ at the ferromagnetic point

To find $\Phi^{(5)}$ in the vicinity of the point $y = 0$ where $2y = 1 - 4w$ requires that we first find a solution to the inhomogeneous equation (2) near $y = 0$. To this end we develop the elliptic

[29] Note a misprint in appendix C in [9], in equation (C7). The right-hand side should in fact be replaced by $8^2(1-2w)^2(1+2w)^2/(1-w)^2/(1-4w)^2$.

integrals in (4) as series in y after which finding a particular integral by series recursion is straightforward. What remains is then the matching of the $\Phi^{(5)}$ series about $w = 0$ to this particular integral plus the 24 homogeneous solution series S_i satisfying $L_{24}(S_i) = 0$. This too is a straightforward exercise although clearly one must pay attention to potential errors arising both from numerical round-off and series length truncation. A useful remark in this regard is that it is very advantageous to work with $\Phi^{(5)}$ series in s rather than $w = s/2/(1+s^2)$. This arises because of the nonlinear relationship $1-4w \approx (1-s)^2/2$ at the ferromagnetic critical point.

To understand the effect of the choice of series variable we note that in general the error resulting from truncation of a series in x at N terms when evaluated at x_m scales roughly as $(x_m/x_s)^N$ where x_s is the distance to the nearest singularity from[30] $x = 0$. As an illustration, from the $\Phi^{(5)}$ series of $N = N_w$ terms and a matching point at $w_m = (1-2y_m)/4$, the series truncation error is $\approx (4w_m)^N = (1-2\,y_m)^N \approx \exp(-2\,N_w\,y_m)$ for small y_m. Now suppose the solution series S_i have been evaluated to $N = N_y$ terms. The solutions with the smallest radius of convergence[31], $y_s = 1/2 - 2w_s \approx 0.183$, determine the truncation error which is $\approx (y_m/0.183)^N \approx \exp(-N_y \ln(0.183/y_m))$. The optimal choice of the matching point is where the errors are roughly equal, i.e.

$$2\,N_w \cdot y_m = N_y \cdot \ln(0.183/y_m). \tag{B.1}$$

With $N_w = 8000$ and $N_y = 800$, condition[32] (B.1) yields $y_m = 0.0577$ with a resulting truncation error $\approx 10^{-400}$. This estimate of 400 digit accuracy is overly optimistic since the $\Phi^{(5)}$ matching requires the evaluation of 23 derivatives which invalidates our assumption of functions all of unit magnitude.

Consider now what happens if the matching is done using $\Phi^{(5)}$ as a series in s of $N = N_s = N_w$ terms. The truncation error is $\approx (s_m)^N \approx (1-2\sqrt{y_m})^N \approx \exp(-2\,N_w\sqrt{y_m})$ for small y_m. The match point choice that replaces (B.1) is

$$2\,N_w \cdot \sqrt{y_m} = N_y \cdot \ln(0.183/y_m), \tag{B.2}$$

and with the same series lengths as above, $y_m = 0.015\,35$ with the resulting truncation error $\approx 10^{-860}$. Again this must be overly optimistic but it does illustrate the dramatic improvement achieved with no extra computational cost.

Our results for the behaviour of $\Phi^{(5)}$ at the ferromagnetic point are as follows: we use throughout the definition $2\,y = 1-4w$ as above. Then, to $O(y^{10})$, the 5-particle contribution is

$$\begin{aligned}
120\,\pi^4 \cdot \Phi^{(5)} &= (\tilde{\chi}^{(1)} - 60\,\tilde{\chi}^{(3)} + 120\,\tilde{\chi}^{(5)}) \cdot \pi^4 \\
&= \ln^4(y/4) \cdot (-5/128 - 25y/512 - 75y^2/512 + 6455y^3/8192 \\
&\quad + 42\,305y^4/16\,384 + 49\,935y^5/8192 + 444\,955y^6/32\,768 \\
&\quad + 64\,409\,075\,y^7/2097\,152 + 306\,977\,235\,y^8/4194\,304 \\
&\quad + 3131\,527\,805y^9/16\,777\,216) \\
&\quad + \ln^3(y/4) \cdot (-235/192 - 1043\,y/512 - 559\,y^2/256 \\
&\quad + 1007\,843\,y^3/1032\,192 + 265\,627\,475\,y^4/22\,708\,224 \\
&\quad + 21\,838\,013\,489\,y^5/590\,413\,824 + 57\,270\,165\,499\,y^6/590\,413\,824 \\
&\quad + 315\,303\,684\,751\,873\,y^7/1284\,740\,481\,024 \\
&\quad + 31\,001\,275\,613\,111\,851\,y^8/48\,820\,138\,278\,912
\end{aligned}$$

[30] We assume for purposes of our qualitative discussion here that all functions are approximately of unit magnitude.
[31] This w_s is the same as that in the discussion leading to and following (9) and (10).
[32] We have reached such values for exact rational coefficient series without too much computing effort.

$$
\begin{aligned}
&+ 336\,259\,319\,399\,213\,305\; y^9/195\,280\,553\,115\,648) \\
&+ \ln^2(y/4) \cdot (-1225/192 + 306\,283\; y/15\,360 - 859\,553\; y^2/215\,040 \\
&- 20\,921\,323\,001\; y^3/433\,520\,640 - 10\,704\,883\,015\,027\; y^4/104\,911\,994\,880 \\
&- 12\,546\,069\,917\,407\,919\; y^5/70\,920\,508\,538\,880 \\
&- 1021\,232\,328\,273\,251\; y^6/3546\,025\,426\,944 \\
&- 1219\,521\,777\,522\,960\,411\,013\; y^7/2623\,491\,451\,870\,248\,960 \\
&- 1428\,705\,274\,437\,356\,046\,530\,557\; y^8/1894\,160\,828\,250\,319\,749\,120 \\
&- 3094\,013\,953\,756\,589\,876\,579\,173\; y^9/2525\,547\,771\,000\,426\,332\,160) \\
&+ \ln(y/4) \cdot \left(\sum_{k=0} C_1(k) \cdot y^k \right) \quad + \sum_{k=-1} C_0(k) \cdot y^k
\end{aligned}
\tag{B.3}
$$

where the constant arrays C_1 and C_0 are only known as floating point values. Truncated values are

$$
\begin{aligned}
C_1(0) &= -24.579\,422\,776\,085\,005\,809\,807\,666\,918\,842\,356\,756\,722\,133\,677\,152\,81\,\cdots \\
C_1(1) &= 100.332\,280\,261\,121\,980\,739\,844\,837\,578\,689\,642\,427\,444\,684\,311\,062\,63\,\cdots \\
C_1(2) &= 153.629\,122\,250\,956\,902\,879\,376\,428\,137\,031\,917\,694\,796\,337\,994\,294\,41\,\cdots \\
C_1(3) &= 96.268\,269\,775\,450\,223\,433\,437\,918\,344\,019\,644\,004\,451\,927\,310\,385\,706\,\cdots \\
C_1(4) &= -93.060\,872\,246\,947\,119\,873\,271\,898\,278\,526\,467\,013\,597\,043\,512\,891\,78\,\cdots \\
C_1(5) &= -520.060\,463\,620\,671\,142\,199\,879\,335\,336\,181\,510\,799\,435\,875\,021\,2855\,\cdots \\
C_1(6) &= -1469.482\,403\,857\,530\,493\,377\,284\,131\,808\,565\,295\,376\,128\,516\,174\,659\,\cdots \\
C_1(7) &= -3700.403\,260\,075\,047\,090\,620\,399\,746\,433\,870\,069\,155\,277\,603\,230\,413\,\cdots \\
C_1(8) &= -9303.076\,094\,072\,409\,117\,026\,994\,688\,305\,643\,439\,126\,188\,417\,678\,639\,\cdots \\
C_1(9) &= -24\,403.451\,159\,562\,421\,438\,882\,358\,591\,269\,422\,381\,164\,661\,467\,183\,68\,\cdots \\
C_0(-1) &= 23.164\,561\,203\,366\,712\,117\,448\,548\,909\,598\,809\,004\,328\,248\,610\,670\,601\cdots \\
C_0(0) &= -117.555\,174\,062\,309\,234\,308\,910\,557\,814\,958\,142\,739\,916\,383\,307\,1621\,\cdots \\
C_0(1) &= 256.411\,511\,499\,496\,232\,579\,281\,003\,142\,934\,611\,918\,878\,090\,562\,349\,18\,\cdots \\
C_0(2) &= 350.915\,859\,067\,901\,015\,292\,198\,391\,166\,198\,006\,918\,129\,639\,776\,119\,99\,\cdots \\
C_0(3) &= 293.478\,939\,167\,681\,863\,066\,232\,363\,606\,680\,739\,151\,434\,085\,014\,857\,41\,\cdots \\
C_0(4) &= 109.389\,343\,633\,237\,837\,047\,699\,483\,772\,407\,094\,363\,666\,814\,669\,906\,51\,\cdots \\
C_0(5) &= -373.928\,805\,577\,364\,161\,339\,869\,290\,516\,560\,018\,802\,311\,340\,499\,2740\,\cdots \\
C_0(6) &= -1561.022\,604\,360\,024\,268\,401\,667\,883\,538\,021\,798\,990\,658\,115\,951\,832\,\cdots \\
C_0(7) &= -4574.248\,865\,947\,314\,517\,242\,004\,568\,453\,306\,010\,341\,709\,549\,402\,520\,\cdots \\
C_0(8) &= -12595.461\,191\,916\,745\,987\,116\,447\,669\,738\,840\,645\,746\,722\,234\,825\,38\,\cdots \\
C_0(9) &= -35224.559\,613\,707\,790\,085\,325\,694\,509\,320\,788\,019\,454\,777\,282\,875\,48\,\cdots
\end{aligned}
\tag{B.4}
$$

while values to about 800 digits can be found in [10]. The 3-particle function, to $O(y^{10})$, is

$$
\begin{aligned}
&-6\,\pi^2 \cdot \Phi^{(3)} = (\tilde{\chi}^{(1)} - 6\,\tilde{\chi}^{(3)}) \cdot \pi^2 = \\
&\quad \ln^2(y/4) \cdot (3/32 - 3y/128 + 9\,y^2/128 + 87\,y^3/2048 \\
&\quad - 39\,y^4/4096 - 423\,y^5/4096 - 555\,y^6/2048 - 301\,647\,y^7/524\,288 \\
&\quad - 1185\,363\,y^8/1048\,576 - 8996\,817\,y^9/4194\,304)
\end{aligned}
$$

$$+\ln(y/4)\cdot(23/32+227\,y/1280-2047\,y^2/4480-88\,949\,y^3/122\,880$$
$$-20\,562\,503\,y^4/18\,923\,520-810\,591\,833\,y^5/492\,011\,520$$
$$-316\,471\,567\,y^6/123\,002\,880-4445\,362\,809\,179\,y^7/1070\,617\,067\,520$$
$$-3085\,016\,829\,083\,y^8/447\,070\,863\,360$$
$$-273\,049\,228\,448\,281\,y^9/23\,247\,684\,894\,720)$$
$$+(41/96-21\,169\,y/38\,400-225\,583\,y^2/3763\,200$$
$$+2621\,231\,y^3/154\,828\,800+15\,533\,081\,173\,y^4/262\,279\,987\,200$$
$$+151\,705\,656\,477\,979\,y^5/1595\,711\,442\,124\,800$$
$$+30\,692\,556\,260\,057\,y^6/265\,951\,907\,020\,800$$
$$+1747\,927\,197\,871\,890\,533\,y^7/19\,676\,185\,889\,026\,867\,200$$
$$-41\,073\,304\,786\,882\,831\,381\,y^8/655\,671\,055\,932\,802\,990\,080$$
$$-776\,816\,275\,820\,131\,600\,824\,097\,y^9/1534\,270\,270\,882\,758\,996\,787\,200)$$
$$+(9\,\psi^{(1)}(1/3)/8-3/2-3\,\pi^2/4)\cdot(1/y-3/2-5y/24-3y^2/8$$
$$-1801\,y^3/3456-1649\,y^4/2304-187\,999\,y^5/186\,624$$
$$-45\,617\,y^6/31\,104-17\,592\,665y^7/7962\,624$$
$$-164\,030\,851\,y^8/47\,775\,744-9425\,604\,977y^9/1719\,926\,784), \qquad \text{(B.5)}$$

where $\psi^{(1)}(x) = \psi'(x)$ is the polygamma function ($\psi(x) = \Gamma'(x)/\Gamma(x)$). The coefficient of $1/y$ in (B.5) was determined by Tracy [16] while the higher order terms are recursively generated from the ODE satisfied [13] by $\Phi^{(3)}$. Finally,

$$\Phi^{(1)} = \tilde{\chi}^{(1)} = 1/(4y) - 1/2. \qquad \text{(B.6)}$$

When these results are combined and expressed in terms of $\tau = (1/s - s)/2$, i.e. $2\,y = \tau^2/(1+\tau^2+\sqrt{1+\tau^2})$, one confirms the low-order result for $\chi^{(5)}$ in equation (48) of [8]. We also confirm the 500 digit expression for the $\chi^{(5)}$ amplitude D_5 given by Bailey *et al* [11]. The explicit connection is

$$D_5 = 52\,\pi^4/5 - 12\,\pi^2\,\psi^{(1)}(1/3) + 16\,\pi^2 + 16\,C_0(-1)/15 \qquad \text{(B.7)}$$

with $C_0(-1)$ given in (B.4).

The 5-particle contribution (B.3) to $O(y^{10})$ can be extended to arbitrary order by a purely local analysis of the ODE (2). This makes (B.3) particularly useful for those problems in which one wants to analytically continue $\Phi^{(5)}$ beyond the ferromagnetic singularity. Of the constants in (B.4), only the 15 values $C_1(0...5)$ and $C_0(-1...7)$ depend directly on solution matching. All higher order coefficients are fixed by recursion relations with rational coefficients. The rationals are quite large, even in the simplest example which reads

$$28\,235\,138\,014\,521\,201\,151\,215\,868\,809\,287\,621\,440\cdot C_1(6) = -\frac{n_1}{d_1}$$
$$-37\,737\,948\,182\,868\,464\,640\,476\,571\,846\,064\,977\,272\,634\,973/1168\,128\cdot C_1(0)$$
$$-2562\,052\,253\,560\,544\,220\,030\,628\,882\,123\,379\,039\,659\,169/86\,528\cdot C_1(1)$$
$$+8187\,149\,177\,658\,628\,591\,337\,051\,452\,109\,148\,482\,747/312\cdot C_1(2)$$
$$+158\,617\,170\,239\,026\,490\,613\,752\,192\,866\,932\,708\,861\,763/1872\cdot C_1(3)$$
$$-140\,695\,455\,847\,623\,820\,725\,386\,797\,097\,484\,481\,682\,383/702\cdot C_1(4)$$
$$+133\,960\,461\,264\,493\,957\,765\,301\,921\,009\,624\,203\,227\cdot C_1(5), \qquad \text{(B.8)}$$

with

$$n_1 = 15\,256\,931\,516\,199\,856\,571\,741\,494\,453\,893\,800\,774\,223\,260\,951\,365\,779\,571\,949,$$
$$d_1 = 31\,365\,304\,106\,405\,068\,800.$$

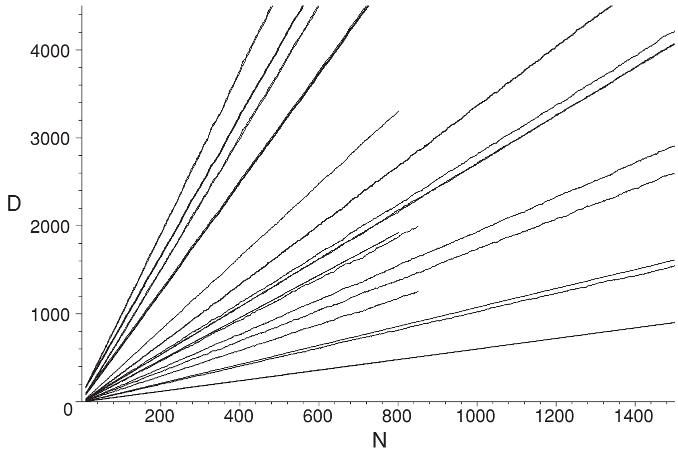

Figure B1. The number of digits D in the denominators of the rational coefficients c_N in series $y^p \ln^q(y/4) \sum c_N y^N$ that arise in ODE solutions around $2y = 1 - 4w = 0$.

The floating point recursion in y is unstable to the extent that relative errors increase by about factor 40 per order in the y series. If one wants, say, 1500 terms, one must start with 2400 digits more than the desired final accuracy. One must also use directly as starting values only the coefficients $C_1(0\ldots5)$ and $C_0(-1\ldots7)$ from (B.4). All other coefficients are to be obtained to the necessary higher accuracy directly from the ODE (2) locally around $y = 0$, now treating the starting values $C_1(0\ldots5)$ and $C_0(-1\ldots7)$ as if exact.

The problem of numerical instability in series generation around rational points can be avoided by generating all series in exact arithmetic. In that case the series coefficients are in general rational and our observation, admittedly rather limited, is that the growth of the denominators is at most exponential. An example of denominator growth is shown in figure B1 for all series in y that we have generated for doing the ferromagnetic point series matching described above. A growth rate faster than exponential would make exact arithmetic series generation impractical so it is of some importance to know whether the observed exponential growth is a general feature. It is reminiscent of G-series [17] but we are unaware of any theorem that guarantees this behaviour for expansions of integrals with algebraic integrands about singular points.

Appendix C. Right division operators for irrational singular points

We wish to determine a right division of a minimum order $L_N(w)$ from solutions identified with the non-rational zeros of a factor $h(w)$ of the head polynomial of $L_N(w)$. Examples of $h(w)$ treated in the text are $1+3w+4w^2$ and the case 3 and 4 polynomials from appendix C in [8]. If such a polynomial has an 'accidental' modulo prime p factorization $h(w) = (w - w_p)\, h_p(w)$, we can still generate series solutions about $x = 0$ where $x = w - w_p$. Furthermore, suppose the solution $S(x)$ has been tested and is found to be annihilated by a minimum order $M < N$ operator. If the code used for the testing is that described in section 3 of [8] the result will be of the form

$$L_M(x) = \sum_{n=0}^{M} f_n(x) \cdot \left(x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\right)^{M-n}, \tag{C.1}$$

where the head polynomial $f_0(x)$ is normalized such that $f_0(x = 0) = 1$. Note that w_p is not the modulo p representation of a root of $h(w)$ and both it and x have no meaning independent of the specific modulo p factorization $h(w) = (w - w_p)h_p(w)$. Consequently there does not exist any exact '$L_M(x)$' to be reconstructed from multiple $L_M(x)$ generated using different primes. Replacing x by $w - w_p$ in (C.1) generates an $L_M(w)$ but only when this is correctly normalized to remove all dependence on the 'accidental' w_p can it be used for a reconstruction of an exact $L_M(w)$. The steps for doing this are as follows.

To begin, note that (C.1) can be expressed as a sum of polynomials times powers of $\mathrm{d}/\mathrm{d}x$. The head polynomial in this representation is clearly $x^M f_0(x)$. The modulo p factorization of this will contain as factors $x^M[h_p(x + w_p)]^K$, typically with $K < M$. These factors can be written as $x^{M-K}[xh_p(x+w_p)]^K = x^{M-K}[h(x+w_p)]^K = (w-w_p)^{M-K}[h(w)]^K$. Dependence on the 'accidental' w_p appears only in the factor x^{M-K} and this can be divided out. So too can the $w = 0$ normalization constant $C = x^K f_0(x)$ evaluated at $x = -w_p$. The result is the right division (modulo p) operator which can be cast into the same form as (C.1), namely

$$L_M(w) = \left.\frac{L_M(x)}{x^{M-K}C}\right|_{x=w-w_p} \quad \text{modulo } p$$
$$= \sum_{n=0}^{M} g_n(w) \cdot \left(w \cdot \frac{\mathrm{d}}{\mathrm{d}w}\right)^{M-n}, \tag{C.2}$$

where each $g_n(w)$ is a polynomial and, by construction, $g_0(w = 0) = 1$. The exact $L_M(w)$ can now be reconstructed from different prime-based versions of (C.2).

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